4.

(a)

$$V_n^2 = k\sum_{j=1}^{n}\left(x_j-\mu_n\right)^2$$

$$E\left(V_n^2\right)=k\sum_{j=1}^{n}E\left[\left(x_j-\mu\right)-\frac{1}{n}\sum_{i=1}^{n}\left(x_i-\mu\right)\right]^2$$

$$=k\sum_{j=1}^{n}E\left\{\left(x_j-\mu\right)^2-\frac{2}{n}\sum_{i=1}^{n}\left(x_j-\mu\right)\left(x_i-\mu\right)+\frac{1}{n^2}\left[\sum_{i=1}^{n}\left(x_i-\mu\right)\right]^2\right\}$$

$$=k\sum_{j=1}^{n}\left[\sigma^2-\frac{2}{n}\sigma^2+\frac{n}{n^2}\cdot\sigma^2\right]=k\cdot n\cdot\frac{n-1}{n}\sigma^2=\sigma^2$$

$$\therefore k=\frac{1}{n-1}$$

(b)

$$E\left|\mu_n-\mu\right|^2=E\left[\frac{1}{n}\sum_{i=1}^{n}\left(x_i-\mu\right)\right]^2=\frac{1}{n^2}\sum_i\sum_j E\left[x_i-\mu\right]\left[x_j-\mu\right].$$

$$=\frac{1}{n^2}\cdot n\sigma^2=\frac{\sigma^2}{n}$$

(c)

$$\therefore E\left|\mu_n-\mu\right|^2=\frac{\sigma^2}{n}=\frac{E\left(V_n^2\right)}{n}\to 0 \quad \text{but } EV_n^2 \not\to 0 \text{ as } n\to\infty$$

5.

(a)

$$f\left(x,y,z\right)=k\left(x+y+z\right) \quad ,\ 0\le x\le 1\ ,\ 0\le y\le 1\ ,\ 0\le z\le 1$$

$$\therefore \int_0^1\int_0^1\int_0^1 k\left(x+y+z\right)dxdydz=1$$

$$\Rightarrow k\int_0^1\int_0^1\left(\frac{1}{2}+y+z\right)dydz=1$$

$$\Rightarrow k\int_0^1\left(z+1\right)dz=1\Rightarrow k\cdot\frac{3}{2}=1\Rightarrow k=\frac{2}{3}$$

(b)

$$f_{X,Z}\left(x,z\right)=\int_0^1\frac{2}{3}\left(x+y+z\right)dy=\frac{2}{3}\left(x+y+\frac{1}{2}\right)\ ,\ \begin{matrix}0\le x\le 1\\ 0\le z\le 1\end{matrix}$$

$$f_X\left(x\right)=\int_0^1\frac{2}{3}\left(x+z+1\right)dz=\frac{2}{3}\left(x+1\right)\ ,\ 0\le x\le 1$$

$$\therefore f_Y\left(y\right)=\frac{2}{3}\left(y+1\right)\ ,\ 0\le y\le 1\ ,\ \text{by similarity}$$

Quiz #6 Sol.

(a) $\int_0^1\int_0^1 f_{X,Y}(x,y)=cdxdy=1.\therefore\ c=1.$

(b) $F_{X,Y}(x,y)=P[X\le x, Y\le y]=\int_0^y\int_0^x f_{X,Y}(x',y')dx'dy'=xy;$

$$F_{X,Y}(x,y)=\begin{cases} xy; & (x,y)\in S \\ 1; & (x,y)\in A \\ y; & (x,y)\in B \\ x; & (x,y)\in C \\ 0; & \text{elsewhere.}\end{cases}$$

(c) $F_X(x)=\int_0^1\int_0^x f_{X,Y}(x',y')dx'dy'=x.$

(d) $F_{X,Y}(x,y)=xy=F_X(x)F_Y(y).X$ and Y are indep.

(e) Let $Z=\max\{X,Y\}$. $F_Z(z)=P[Z\le z]=P[X\le z, Y\le z]=F_{X,Y}(z,z)=z^2$.

$$F_Z(z)=\begin{cases}0, & z<0;\\ z^2, & 0\le z\le 1;\\ 1, & z>1.\end{cases}\quad \therefore\ f_Z(z)=\begin{cases}0, & z<0;\\ 2z, & 0\le z\le 1;\\ 0, & z>1.\end{cases}$$

(a) Find $f_Y(y\,|\,x)$ by differentiating $P[Y\le y\,|\,X=x]$.

$$f_Y(y\,|\,x)=\frac{d}{dy}P[Y\le y\,|\,X=x]=\begin{cases}\frac{1}{x}, & 0\le y\le x;\\ 0, & y>x.\end{cases}$$

(b) Find $E[Y\,|\,X=x]$ from (a).

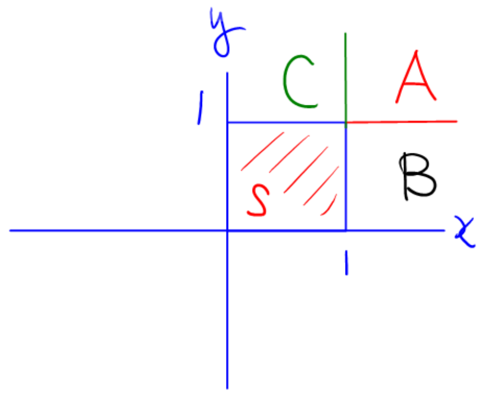
$$E[Y\,|\,X=x]\triangleq\int_0^x yf_Y(y\,|\,x)dy=\int_0^x y\frac{1}{x}dy=\frac{x}{2}$$

(c) Plot the sample space of (X,Y)

(d) Find $f_{X,Y}(x,y)=f_Y(y\,|\,x)f_X(x)$ from (a).Verify it can be a pdf.

$$f_{X,Y}(x,y)=\begin{cases}\frac{1}{x}, & 0\le y\le x\le 1;\\ 0, & \text{otherwise.}\end{cases}$$

$$\int_0^1\int_0^x f_{X,Y}(x,y)dydx=\int_0^1\int_0^x\frac{1}{x}dydx=\int_0^1 1dx=1.$$



(e) Find $E[XY]$ by the definition from (d).

$$E[XY]=\int_0^1\int_0^x xyf_{X,Y}(x,y)dydx=\int_0^1\int_0^x xy\frac{1}{x}dydx=\frac{1}{6}$$

(f) Find $E[XY]$ by $E[XY]=E[E[XY\,|\,X]]$ from (b).

$$E[E[XY\,|\,X]]=E[XE[Y\,|\,X]]=E\left[X\frac{X}{2}\right]=\frac{1}{2}\frac{1}{3}=\frac{1}{6}$$